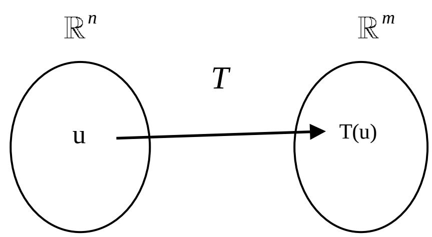
Linear Transformations and Matrices

Definition

Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a mapping or transformation from an n^{th} dimensional to an m^{th} dimensional vector space. The **transformation** T is called **linear** if the following two conditions are true for any constant c and all "vectors" u and v in $\mathbb{R}^n$.

(i) $T(u+v) = T(u) + T(v)$

(ii) $T(c \cdot u) = c \cdot T(u)$



Example 1

Show that the following transformation is linear.

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-3y \end{bmatrix}$$

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be vectors in $\mathbb{R}^2$ and c be a constant.

(i) $\vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$

$$T(\vec{u} + \vec{v}) = T\begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 + u_2 + v_2 \\ u_1 + v_1 - 3(u_2 + v_2) \end{pmatrix} = \begin{pmatrix} u_1 + v_1 + u_2 + v_2 \\ u_1 + v_1 - 3u_2 - 3v_2 \end{pmatrix}$$

$$= \begin{pmatrix} u_1 + u_2 \\ u_1 - 3u_2 \end{pmatrix} + \begin{pmatrix} v_1 + v_2 \\ v_1 - 3v_2 \end{pmatrix} = T(\vec{u}) + T(\vec{v})$$

(ii) $c\vec{u} = c\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix}$

$$T(c\vec{u}) = T\begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix} = \begin{pmatrix} cu_1 + cu_2 \\ cu_1 - 3cu_2 \end{pmatrix} = c\begin{pmatrix} u_1 + u_2 \\ u_1 - 3u_2 \end{pmatrix} = cT(\vec{u})$$

Example 2

Show that the following transformation is **not** linear.

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+1 \\ 2y \end{bmatrix}$$

$$T(c\vec{u}) = T\begin{pmatrix} cu_1 \\ cu_2 \end{pmatrix} = \begin{pmatrix} cu_1 + 1 \\ 2cu_2 \end{pmatrix} \neq cT(\vec{u}) \quad \left[cT(\vec{u}) = c\begin{pmatrix} u_1 + 1 \\ 2u_2 \end{pmatrix} \right]$$

Matrix Representation of Linear Transformations

To find the **standard matrix representation** of any linear transformation T, apply T on the identity vectors. In other words,

$$T = \begin{bmatrix} T\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \cdots & T\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \end{bmatrix}$$

In example 1 from the previous page,

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-3y \end{bmatrix}$$

$$T = \begin{bmatrix} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

Later, we will call $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ the standard basis in $\mathbb{R}^2$.

Homework

1. Consider the linear transformation given by

$$T\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}x_1-x_2\\x_2-x_3\\x_3-x_4-x_1\end{pmatrix}$$

   a. Find the matrix representation of T.

   b. Compute $T\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ using the formula of T.

   c. Compute $T\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ using the matrix representation of T.

2. Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be a linear transformation such that its matrix representation is

$$T=\begin{bmatrix}-2&1&-2\\1&1&0\\3&-1&6\end{bmatrix}.$$

   Find a formula for $T\begin{bmatrix}x\\y\\z\end{bmatrix}$.

TABLE 1 Reflections

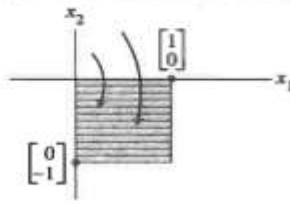
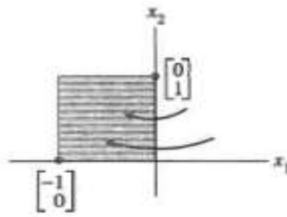
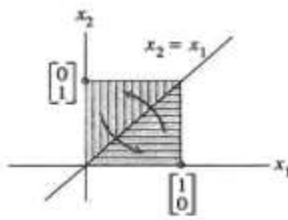
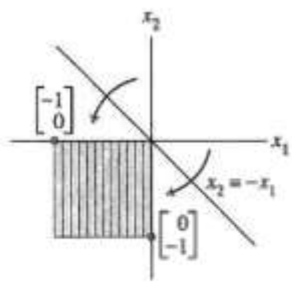
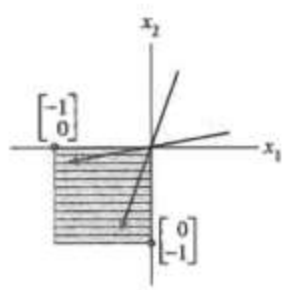
Transformation	Image of the Unit Square	Standard Matrix
Reflection through the x_1-axis		$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$
Reflection through the x_2-axis		$\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$
Reflection through the line $x_2 = x_1$		$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$
Reflection through the line $x_2 = -x_1$		$\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}$
Reflection through the origin		$\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$

TABLE 2 Contractions and Expansions

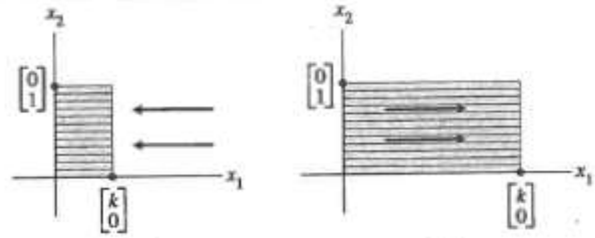
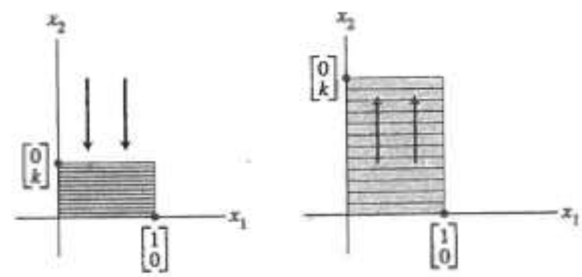
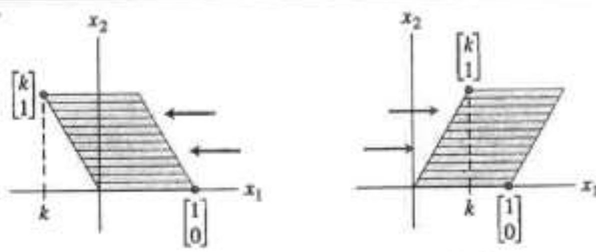
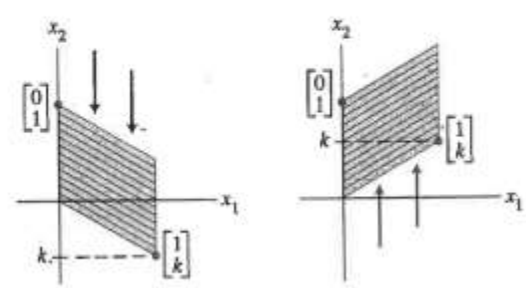
Transformation	Image of the Unit Square	Standard Matrix
Horizontal contraction and expansion	 $0 < k < 1$; $k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion	 $0 < k < 1$; $k > 1$	$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

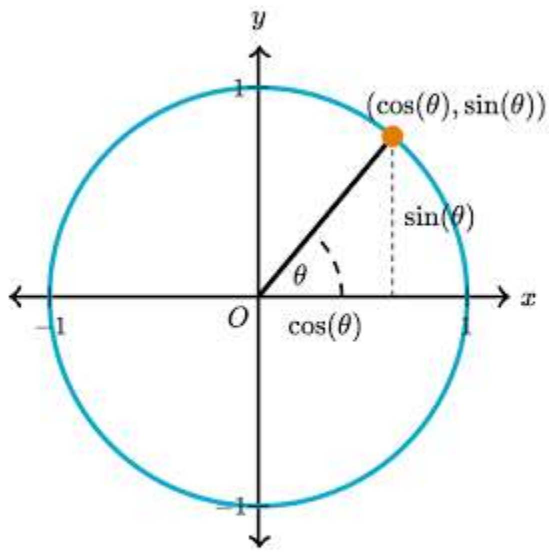
TABLE 3 Shears

Transformation	Image of the Unit Square	Standard Matrix
Horizontal shear	 $k < 0$; $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear	 $k < 0$; $k > 0$	$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Rotations

Counterclockwise

Equation of a unit circle centered at (0, 0): $x^2 + y^2 = 1$



Using the fact that the equation of the unit circle centered at $(0,0)$ can be written in the form $\cos^2\theta + \sin^2\theta = 1$, the matrix representation $\left(R_{cc}\right)$ of a counterclockwise rotation by θ in $\mathbb{R}^2$ can be easily derived to be

$$R_{cc} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

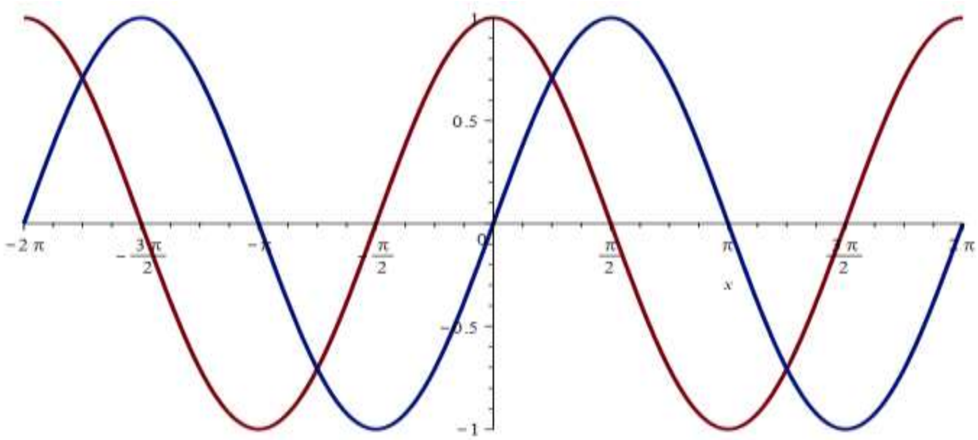
$$R_{cc} = \left[R_{cc}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad R_{cc}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \begin{bmatrix} \cos\theta & \cos\left(\frac{\pi}{2}+\theta\right) \\ \sin\theta & \sin\left(\frac{\pi}{2}+\theta\right) \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos\frac{\pi}{2}\cdot\cos\theta - \sin\frac{\pi}{2}\cdot\sin\theta \\ \sin\theta & \sin\frac{\pi}{2}\cdot\cos\theta + \sin\theta\cdot\cos\frac{\pi}{2} \end{bmatrix}$$

Note: It's a common practice to use the same variable for the transformation as a mapping, and the matrix representing the same transformation.

Clockwise

For clockwise rotations $\left(R_c\right)$ by θ in $\mathbb{R}^2$, just replace θ by $-\theta$ in R_{cc} and we get

$$R_c = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$



Homework

1. Compute $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} =$

2. What can you say about the columns of the matrix R_{cc} (as vectors)?

3. Find a formula for R_{cc}^2.

$$R_{cc}^2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} =$$